Fibonacci Worksheet

From Fibonacci's *Liber Abaci*

1. Two birds start flying from the tops of two towers 50 feet apart; one tower is 30 feet high and the other 40 feet high. Starting at the same time and flying at the same rate, the birds reach a fountain between the bases of the towers at the same moment. How far is the fountain from each tower?

2. A merchant doing business in Lucca doubled his money there and then spent 12 denarii. On leaving, he went to Florence, where he also doubled his money and spent 12 denarii. Returning home to Pisa, he there doubled his money and again spent 12 denarii, nothing remaining. How much did he have in the beginning?

3. Three men, each having denarii, found a purse containing 23 denarii. The first man said to the second, "If I take this purse, I will have twice as much as you." The second said to the third, "If I take this purse, I will have three times as much as you." The third man said to the first, "If I take this purse, I will have four times as much as you." How many denarii did each man have?

4. (The Lion and the Pit: *De leone quite erat in puteo*). A pit was 50 handbreadths in depth: A lion climbed up the pit 1/7 handbreadth every day and fell back 1/9 handbreadth. How long would it take him to get out of the pit?

5. A man whose end was approaching summoned his sons and said: "Divide my money as I shall prescribe." To his eldest son, he said, "You are to have 1 bezant and a seventh of what is left." To his second son he said, "Take 2 bezants and a seventh of what remains." To the third son, "You are to take 3 bezants and a seventh of what is left." Thus he gave each son 1 bezant more than the previous son and a seventh of what remained, and to the last son all that was left. After following their father's instructions with care, the sons found that they had shared their inheritance equally. How many sons were there, and how large was the estate?

6. Find a number that is divisible by 7 and which gives a remainder of 1 when divided by 2, 3, 4, 5, or 6.

7. Find a number that is a multiple of 7 having remainders of 1, 2, 3, 4 and 5 when divided by 2, 3, 4, 5 and 6.

From Fibonacci's *Liber Quadratorum*

8. Find a square number which when increased or decreased by 5 gives a square number.

9. Given the squares of three successively odd numbers, show that the largest square exceeds the middle square by eight more than the middle square exceeds the smallest.

10. Assuming that x and y are integers:
    a. Find a number of the form

    $$4xy(x+y)(x-y)$$

    That is divisible by 5, the quotient being a square.

    b. Prove that if $x + y$ is even, then the product $xy(x + y)(x - y)$ is divisible by 24, and that without this restriction, $4xy(x + y)(x - y)$ is divisible by 24. [*Hint*: Consider that any integer is of the form $3k$, $3k + 1$, and $3k + 2$ in showing that 3 divides $xy(x + y)(x - y)$. Similarly, because any integer is of the form $8k$, $8k + 1$, $8k + 2$, …, or $8k + 7$, then 8 divides $xy(x + y)(x - y)$.]

11. Solve in rational numbers the pair of equations: $x^2 + x = u^2$, $x^2 - x = v^2$.

12. Find
    a. A square number such that when twice its root is added to it or subtracted from it, one obtains other square numbers. In other words, solve a problem of the type $x^2 + 2x = u^2$, $x^2 - 2x = v^2$ in rational numbers.

    b. Three square numbers such that the addition of the first and second, and also the addition of all three squares, produces square numbers. In other words, solve a problem of the type $x^2 + y^2 = u^2$ and $x^2 + y^2 + z^2 = v^2$ in rational numbers. [*Hint*: Let x and y be two relatively prime integers such that $x^2 + y^2$ equals a square, say $x^2 + y^2 = u^2$. Now note the identity

    $$u^2 + \left(\frac{u^2-1}{2}\right)^2 = \left(\frac{u^2+1}{2}\right)^2 .]$$

13. Fibonacci proved that if the sum of two consecutive integers is a square (that is, if $n + (n + 1) = u^2$ for some u), then the square of the larger integer will equal the sum of two nonzero squares. Verify this result and furnish several numerical examples.

14. Show that the identity
    $$\left(a^2 + b^2\right)\left(c^2 + d^2\right) = \left(ac + bd\right)^2 + \left(ad - bc\right)^2$$
    $$= \left(ad + bc\right)^2 + \left(ac - bd\right)^2$$
    is true. Then use it to show that $65 = 5\times13$, $130 = 10\times13$ and $481 = 13\times37$ can be written as the sum of two squares in two different ways.

From Fibonacci's *Flos*

15. Solve $x^3 + 2x^2 + 10x = 20$.

16. Three men owned some money, their shares being 1/2, 1/3, and 1/6. Each took some money at random until none was left. The first man then returned 1/2 of what he had taken, the second 1/3 and the third 1/6. When the money now in the pile was divided equally among the men, each possessed what he was entitled to. How much money was in the original store, and how much did each man take?

Miscellaneous Problems

17. The Lucas sequence, named for French mathematician Edouard Lucas (1842-1891), uses the same recurrence equation as the Fibonacci sequence $L(n + 2) = L(n + 1) + L(n)$ but starts with initial conditions $L(0) = 2$ and $L(1) = 1$. Write out the first ten (10) Lucas numbers.

18. If $F(n)$ represents the n-th Fibonnaci number and $L(n)$ represents the n-th Lucas number, show that the following relationships are true:
    a. $F(n + 3) + F(n) = 2F(n + 2)$
    b. $F(n + 3) - F(n) = 2F(n + 1)$
    c. $F(n + 4) + F(n) = 3F(n + 2)$
    d. $F(n + 4) - F(n) = L(n + 2)$
    e. $F(n + 6) + F(n) = 2L(n + 3)$
    f. $F(n + 6) - F(n) = 4F(n + 3)$

19. Let $x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}$ show that $x = \dfrac{1 + \sqrt{5}}{2}$.

20. Let $x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$ show that $x = \dfrac{1 + \sqrt{5}}{2}$.