




Europe Smells the Coffee

Chapter 6

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Boethius (480 – 524)

- Boethius became an orphan when he was seven years old.
- He was extremely well educated.
- Boethius was a philosopher, poet, mathematician, statesman, and (perhaps) martyr.



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Boethius (480 – 524)

- He is best known as a translator of and commentator on Greek writings on logic and mathematics (Plato, Aristotle, Nichomachus).
- His mathematics texts were the best available and were used for many centuries at a time when mathematical achievement in Europe was at a low point.
- Boethius' Arithmetic taught medieval scholars about Pythagorean number theory.

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Boethius (480 – 524)

- This shows Boethius calculating with Arabic numerals competing with Pythagoras using an abacus.
- It is from G. Reisch, Margarita Philosophica (1508).



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Boethius (480 – 524)

- Boethius was a main source of material for the quadrivium, which was introduced into monasteries and consisted of arithmetic, geometry, astronomy, and the theory of music.
- Boethius wrote about the relation of music and science, suggesting that the pitch of a note one hears is related to the frequency of sound.

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Boethius (480 – 524)

- One of the first musical works to be printed was Boethius's *De institutione musica*, written in the early sixth century.
- It was for medieval authors, from around the ninth century on, the authoritative document on Greek music-theoretical thought and systems.
- For example, Franchino Gaffurio in *Theorica musica* (1492) acknowledged Boethius as the authoritative source on music theory.

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Boethius (480 – 524)

- His writings and translations were the main works on logic in Europe becoming known collectively as *Logica vetus*.
- Boethius' best-known work is the *Consolations of Philosophy* which was written while he was in prison.
- It looked at the "questions of the nature of good and evil, of fortune, chance, or freedom, and of divine foreknowledge."

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Gregorian Chant

- The term Gregorian chant is named after Pope Gregory I (590–604 AD).
- He is credited with arranging a large number of choral works, which arose in the early centuries of Christianity in Europe.
- Gregorian chant is monophonic, that is, music composed with only one melodic line without accompaniment.

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Gregorian Chant

- As with the melodies of folk music, the chants probably changed as they were passed down orally from generation to generation.
- Polyphony is music where two or more melodic lines are heard at the same time in a harmony.
- Polyphony didn't exist (or it wasn't on record) until the 11th century.

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Polyphony

- Although the majority of medieval polyphonic works are anonymous - the names of the authors were either not preserved or simply never known - there are some composers whose work was so significant that their names were recorded along with their work.
 - Hildegard von Bingen (1098 - 1179)
 - Perotin (1155 - 1377)
 - Guillaume de Machau (1300 - 1377)
 - John Dunstable (1385 - 1453)

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The Dark Ages

- The Dark Ages, formerly a designation for the entire period of the Middle Ages, now refers usually to the period c.450–750, also known as the Early Middle Ages.
- Medieval Europe was a large geographical region divided into smaller and culturally diverse political units that were never totally dominated by any one authority.

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The Middle Ages

- With the collapse of the Roman Empire, Christianity became the standard-bearer of Western civilization.
- The papacy gradually gained secular authority; monastic communities had the effect of preserving antique learning.
- By the 8th century, culture centered on Christianity had been established; it incorporated both Latin traditions and German institutions.

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The Middle Ages

- The empire created by Charlemagne illustrated this fusion.
- However, the empire's fragile central authority was shattered by a new wave of invasions.
- Feudalism became the typical social and political organization of Europe.
- The new framework gained stability from the 11th century, as the invaders became Christian.

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The High Middle Ages

- As Europe entered the period known as the High Middle Ages, the church became the unifying institution.
- Militant religious zeal was expressed in the Crusades.
- Security and prosperity stimulated intellectual life, newly centered in burgeoning universities, which developed under the auspices of the church.

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The High Middle Ages

- From the Crusades and other sources came contact with Arab culture, which had preserved works of Greek authors whose writings had not survived in Europe.
- Philosophy, science, and mathematics from the Classical and Hellenistic periods were assimilated into the tenets of the Christian faith and the prevailing philosophy of scholasticism.

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The High Middle Ages

- Christian Europe finally began to assimilate the lively intellectual traditions of the Jews and Arabs.
- Translations of ancient Greek texts (and the fine Arabic commentaries on them) into Latin made the full range of Aristotelean philosophy available to Western thinkers.
- Aristotle, long associated with heresy, was adapted by St. Thomas Aquinas to Christian doctrine.

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The High Middle Ages

- Christian values pervaded scholarship and literature, especially Medieval Latin literature, but Provencal literature also reflected Arab influence, and other flourishing medieval literatures, including German, Old Norse, and Middle English, incorporated the materials of pre-Christian traditions.

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Thomas Aquinas (1225 – 1272)

- St. Thomas Aquinas was an Italian philosopher and theologian, Doctor of the Church, known as the Angelic Doctor.
- He is the greatest figure of scholasticism - philosophical study as practiced by Christian thinkers in medieval universities.



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Thomas Aquinas (1225 – 1272)

- He is one of the principal saints of the Roman Catholic Church, and founder of the system declared by Pope Leo XIII to be the official Catholic philosophy.
- St. Thomas Aquinas held that reason and faith constitute two harmonious realms in which the truths of faith complement those of reason; both are gifts of God, but reason has an autonomy of its own.

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Thomas Aquinas (1225 – 1272)

- Aquinas's unfinished *Summa Theologica* (1265-1273) represents the most complete statement of his philosophical system.
- The sections of greatest interest include his views on the nature of god, including the five ways to prove god's existence, and his exposition of natural law.

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Natural Law

- Belief that the principles of human conduct can be derived from a proper understanding of human nature in the context of the universe as a rational whole.
- Aquinas held that even the divine will is conditioned by reason.
- Thus, the natural law provides a non-revelatory basis for all human social conduct.

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The Existence of God

- Attempts to prove the existence of god have been a notable feature of Western philosophy.
 - The cosmological argument
 - The ontological argument
 - The teleological argument
 - The moral argument
- The most serious atheological argument is the problem of evil.

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The Cosmological Argument

- An attempt to prove the existence of god by appeal to contingent facts about the world.
- The first of Aquinas's five ways (borrowed from Aristotle's *Metaphysics*), begins from the fact that something is in motion, since everything that moves must have been put into motion by something else but the series of prior movers cannot extend infinitely, there must be a first mover (which is god).

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The Ontological Argument

- Ontological arguments are arguments, for the conclusion that God exists, from premises which are supposed to derive from some source other than observation of the world - e.g., from reason alone.
- Ontological arguments are arguments from nothing but analytic, a priori and necessary premises to the conclusion that God exists.
- St. Anselm of Canterbury claims to derive the existence of God from the concept of a "being than which no greater can be conceived."

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The Ontological Argument

- St. Anselm reasoned that, if such a being fails to exist, then a greater being – namely, a being than which no greater can be conceived, and which exists – can be conceived.
- But this would be absurd, nothing can be greater than a being than which no greater can be conceived.
- So a being than which no greater can be conceived, that is, God, exists.

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The Ontological Argument

- In the 17th century, Rene Descartes endorsed a different version of this argument.
- In the early 18th century, Gottfried Leibniz attempted to fill what he took to be a shortcoming in Descartes' view.
- Recently, Kurt Gödel, best known for his incompleteness theorems, sketched a revised version of this argument.

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The Teleological Argument

- Based upon an observation of the regularity or beauty of the universe.
- Employed by Marcus Cicero (106-43 BC), Aquinas, and William Paley (1743-1805), the argument maintains that many aspects of the natural world exhibit an orderly and purposeful character that would be most naturally explained by reference to the intentional design of an intelligent creator.

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The Teleological Argument

- Let X represent a given species of animal or for a particular organ (e.g. the eye) or a capability of a given species:
 - X is very complicated and/or purposeful.
 - The existence of very complex and/or purposeful things is highly improbable, and thus their existence demands an explanation.
 - The only reasonable explanation for the existence of X is that it was designed and created by an intelligent, sentient designer.

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The Teleological Argument

- X was not designed or created by humans, or any other Earthly being.
- Therefore, X must have been designed and created by a non-human but intelligent and sentient artificer.
- In particular, X must have been designed and created by God.
- Therefore God must exist.
- This argument is very popular today and it is at the core of scientific creationism.

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The Teleological Argument

- Most biologists support the standard theory of biological evolution, i.e., they reject the third premise.
- In other words, Darwin's theory of natural selection offers an alternative, non-teleological account of biological adaptations.
- In addition, anyone who accepts this line of argument but acknowledges the presence of imperfection in the natural order is faced with the problem of evil.

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The Moral Argument

- An attempt to prove the existence of god by appeal to presence of moral value in the universe.
 - There is a universal moral law.
 - If there is a universal moral law, then there must be a universal moral lawgiver.
 - Therefore, there must be God.
- Man is an intelligent creature having a conscience which is based upon an innate moral code.
- This natural law requires a Law-Giver.

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The Moral Argument

- Society with its various forms of government, recognizes the concepts of right and wrong.
- Where does this uniform impulse come from, if not from God.
- The fourth of Aquinas's five ways concludes that god must exist as the most perfect cause of all lesser goods.
- Immanuel Kant argued that postulation of god's existence is a necessary condition for our capacity to apply the moral law.

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The Problem of Evil

- Bad things happen.
- Whether they are taken to
 - flow from the operation of the world ("natural evil"),
 - result from deliberate human cruelty ("moral evil"), or simply
 - correlate poorly with what seems to be deserved ("non-karmic evil").
- Such events give rise to basic questions about whether or not life is fair.

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The Problem of Evil

- The presence of evil in the world poses a special difficulty for traditional theists.
- Since an omniscient god must be aware of evil, an omnipotent god could prevent evil, and a benevolent god would not tolerate evil, it should follow that there is no evil.
- Yet there is evil, from which atheists conclude that there is no omniscient, omnipotent, and benevolent god.

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The Problem of Evil

- The most common theistic defense against the problem, propounded (in different forms) by both Augustine and **Leibniz**, is to deny the reality of evil by claiming that apparent cases of evil are merely parts of a larger whole that embodies greater good.
- More recently, some have questioned whether the traditional notions of omnipotence and omniscience are coherent.

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Pascal's Wager

- Blaise Pascal (1623 - 1662)
 - "It makes more sense to believe in God than to not believe. If you believe, and God exists, you will be rewarded in the afterlife. If you do not believe, and He exists, you will be punished for your disbelief. If He does not exist, you have lost nothing either way."
- It amounts to hedging your bets.

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The Atheist's Wager

- "It is better to live your life as if there are no Gods, and try to make the world a better place for your being in it. If there is no God, you have lost nothing and will be remembered fondly by those you left behind. If there is a benevolent God, He will judge you on your merits and not just on whether or not you believed in Him."

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Fibonacci (1170-1250)

- Fibonacci was born in Italy but was educated in North Africa.
- Fibonacci was taught mathematics in Bugia and traveled widely with his father.



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Fibonacci (1170-1250)

- He recognized the enormous advantages of the mathematical systems used in the countries they visited.
- Fibonacci ended his travels around the year 1200 and returned to Pisa.
- He wrote a number of important texts, including Liber abaci, Practica geometriae, Flos, and Liber quadratorum.

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Fibonacci (1170-1250)

- Johannes of Palermo, a member of the Holy Roman emperor Frederick II's court, presented a number of problems as challenges to Fibonacci.
- Fibonacci solved three of them and put his solutions in Flos.
- Liber abaci, published in 1202, was based on the arithmetic and algebra that Fibonacci accumulated during his travels.

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Fibonacci's Liber Abaci

- It introduced the Hindu-Arabic place-valued decimal system and the use of Arabic numerals into Europe.
- The second section of Liber abaci contains a large number of problems about the price of goods, how to calculate profit on transactions, how to convert between the various currencies in use at that time, and problems which had originated in China.

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Fibonacci's Liber Abaci

- A problem in the third section of Liber abaci led to the introduction of the **Fibonacci numbers** and the **Fibonacci sequence**.
- "A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?"

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Fibonacci Sequence

- The resulting sequence is:
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...
- Notice that each number after the first two (1 and 1) is the sum of the two preceding numbers.
- That is, 13 is the sum of 8 and 5.
- 55 is the sum of 34 and 21.

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Fibonacci Numbers

- The numbers in the sequence are called Fibonacci numbers.
- We call the numbers the terms of the sequence.
- Each term can be denoted using subscripts that identify the order in which the terms appear.
- We denote the Fibonacci numbers by $u_1, u_2, u_3, \dots$, and the n -th term is denoted u_n .

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Fibonacci Numbers

- So $u_1=1, u_2=1, u_3=2, u_4=3, u_5=5$, etc.
- Note that $u_3=u_1+u_2$ and $u_4=u_2+u_3$ and $u_5=u_3+u_4$ and this pattern continues for all terms after the first two.
- Mathematicians write this sequence by stating the initial conditions, $u_1=1$ and $u_2=1$, and using a recursive relation which says, in an equation, that each term is the sum of its two predecessors.

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Fibonacci Sequence

- In this case the recurrence equation is given at the right.
- The last equation in the box says that the $(n+2)$ -nd term is the sum of the n -th term and the $(n+1)$ -st term.

$$\begin{aligned} u_1 &= 1 \\ u_2 &= 1 \\ u_{n+2} &= u_n + u_{n+1} \end{aligned}$$

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Fibonacci Sequence

- When $n = 1$ for example, this says that the third term, u_3 , is the sum of the first term, u_1 , and the second term, u_2 , which is, of course, correct.

$$\begin{aligned} u_1 &= 1 \\ u_2 &= 1 \\ u_{n+2} &= u_n + u_{n+1} \end{aligned}$$

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Fibonacci Sequence

- Fibonacci's sequence of numbers occurs in many places including
 - Pascal's triangle, the binomial formula, probability, the golden ratio, the golden rectangle, plants and nature, and on the piano keyboard, where one octave contains 2 black keys in one group, 3 black keys in another, 5 black keys all together, 8 white keys and 13 keys in total.

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Fibonacci Sequence

- Note that the Fibonacci numbers grow without bound, that is, they become arbitrarily large, in other words, they go to infinity.
- In fact the sixtieth term is $u_{60} = 1,548,008,755,920$.
- And $u_{88} = 1,100,087,778,366,101,931$.
- Wow!

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Fibonacci Sequence

- Consider the sequence of ratios $1/1, 2/1, 3/2, 5/3, 8/5, 13/8, 21/13, \dots$ formed by dividing each term by the one before it.
- Instead of consistently getting larger, they alternate between growing and shrinking!
- In decimal the sequence is $1, 2, 1.5, 1.666\dots, 1.6, 1.625, 1.615384\dots$, etc.
- It turns out that the sequence of ratios approaches a single target which we can readily calculate using a clever argument.

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Fibonacci Sequence

- Let's call the target (or limit as mathematicians say) L .
- Let's denote the n -th ratio by R_n .
- In other words,

$$R_n = \frac{u_{n+1}}{u_n}$$

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Fibonacci Sequence

- Dividing Fibonacci's recurrence equation by u_{n+1} gives

$$\frac{u_{n+2}}{u_{n+1}} = \frac{u_n}{u_{n+1}} + \frac{u_{n+1}}{u_{n+1}}$$

which is equal to

$$R_{n+1} = \frac{1}{R_n} + 1$$

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Fibonacci Sequence

- If we let n approach infinity, then the last equation becomes

$$L = \frac{1}{L} + 1$$

which is equivalent to

$$L^2 = 1 + L$$

$$L^2 - L - 1 = 0$$

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Fibonacci Sequence

- Using the quadratic formula, we get that

$$L = \frac{1 + \sqrt{5}}{2}$$

- This is called the golden ratio, and it was known to the ancient Greeks as the most pleasing ratio of the length of a rectangular painting frame to its width.

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The Golden Ratio ϕ

- Consider a rectangle whose width is 1 and whose length is L .
- They assumed that the perfect or golden rectangle has the property that the removal of a square from it leaves a (smaller) rectangle that is similar to the original one.

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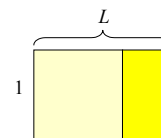
55



The Golden Ratio ϕ

- So the yellow rectangle whose width is $L - 1$ and whose length is 1 is similar to the original.
- So

$$\frac{L}{1} = \frac{1}{L-1}$$



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The Golden Ratio ϕ

- Cross multiplying gives

$$L \times (L - 1) = 1 \times 1$$

$$L^2 - L = 1$$

$$L^2 - L - 1 = 0$$

- This is the quadratic equation of the Fibonacci ratios. Wow!

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The Golden Ratio ϕ

- The face of the Parthenon in Athens has been seen as a golden rectangle and so have many other facades in Greek and Renaissance architecture.
- The golden ratio appears in many strange places in both the natural world and the human world of magnificent artistic and scientific achievements.

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The Golden Ratio ϕ

- Psychologists have shown that the golden ratio subconsciously affects many of our choices, such as where to sit as we enter a large auditorium, where to stand on a stage when we address an audience, etc. etc.
- See Ron Knott's [Fibonacci Numbers and the Golden Section](#).

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The Golden Ratio ϕ

- Euclid, in The Elements (Book VI, Proposition 30), says that the line AB is **divided in extreme and mean ratio** by C if $AB:AC = AC:CB$.
- We would call it "finding the golden section C point on the line".



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The Golden Ratio ϕ

- Euclid used this phrase to mean the ratio of the smaller part of this line, CB to the larger part AC (ie the ratio CB/AC) is the SAME as the ratio of the larger part, AC to the whole line AB (ie is the same as the ratio AC/AB).
- If we let the line AB have unit length and AC have length x (so that CB is then just $1-x$) then the definition means that

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The Golden Ratio ϕ

$$\frac{CB}{AC} = \frac{AC}{AB} \Rightarrow \frac{1-x}{x} = \frac{x}{1}$$

- Solving gives $x = \frac{-1+\sqrt{5}}{2}$
- The golden ratio is $1/x = 1.61803398\dots$
- See [The Golden Ratio](#)

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Fibonacci's Liber Abaci

- Other types of problems in the third section of Liber abaci include:
 - A spider climbs so many feet up a wall each day and slips back a fixed number each night, how many days does it take him to climb the wall.
 - A hound whose speed increases arithmetically chases a hare whose speed also increases arithmetically, how far do they travel before the hound catches the hare.
 - Calculate the amount of money two people have after a certain amount changes hands and the proportional increase and decrease are given.

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Fibonacci's Liber Abaci

- There are also problems involving perfect numbers, the Chinese remainder theorem and problems involving the summing arithmetic and geometric series.
- In the fourth section, he deals with irrational numbers both with rational approximations and with geometric constructions.

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Fibonacci's Practica geometriae

- It contains a large collection of geometry problems arranged into eight chapters with theorems based on Euclid's Elements and On Divisions.
- It includes practical information for surveyors, including a chapter on how to calculate the height of tall objects using similar triangles.
- Included is the calculation of the sides of the pentagon and the decagon from the diameter of circumscribed and inscribed circles.
- The inverse calculation is also given.

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Fibonacci's Flos

- In it he gives an accurate approximation to a root of $10x + 2x^2 + x^3 = 20$, one of the problems that he was challenged to solve by Johannes of Palermo.
- Johannes of Palermo took this problem from Omar Khayyam's algebra book where it is solved by means of the intersection a circle and a hyperbola.
- Fibonacci proves that the root of the equation is neither an integer nor a fraction, nor the square root of a fraction.

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Fibonacci's Flos

- Without explaining his methods, Fibonacci then gives the approximate solution in sexagesimal notation as $(1;22,7,42,33,4,40)_{60}$ (This is $1 + 22/60 + 7/60^2 + 42/60^3 + \dots$).
- This converts to the decimal 1.3688081075 which is correct to nine decimal places.
- This is a truly remarkable achievement.

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Fibonacci's Liber quadratorum

- Liber quadratorum, written in 1225, is Fibonacci's most impressive piece of work.
- It is a number theory book.
- In it, he provides a method for finding Pythagorean triples.
- Fibonacci first notes that any square number is the sum of consecutive odd numbers.
- For example, $1+3+5+7+9=25=5^2$!

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Fibonacci and Pythagorean Triples

- Recall that any odd number is of the form $2n+1$, for some integer n .
- He noticed that the formula

$$n^2 + (2n+1) = (n+1)^2$$
 implies that a square plus an odd number equals the next higher square.
- For example, if $n=2$ then we get the equation $2^2 + (2 \times 2 + 1) = (2+1)^2$ which is equivalent to $2^2 + 5 = 3^2$

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Fibonacci and Pythagorean Triples

- What does this have to do with Pythagorean triples?
- Well, Fibonacci said, "Ifa the odd numba $2n+1$ isa squara then you hava Pythagorean triple!"
- For example, if $n=4$, then $2n+1=2(4)+1=9$ then we get the equation $4^2 + (2 \times 4 + 1) = (4+1)^2$ which is equivalent to $4^2 + 3^2 = 5^2$
- Which gives us the famous (3, 4, 5) Pythagorean triple.

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Fibonacci's Liber quadratorum

- Fibonacci also proves many interesting number theory results including the fact that there is no x, y such that

$$x^2 + y^2 \text{ and } x^2 - y^2$$
 are both squares.
- He also proves that $x^4 - y^4$ cannot be a square.

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Fibonacci's Liber quadratorum

- He defined the concept of a **congruum**, a number of the form $ab(a+b)(a-b)$, if $a+b$ is even, and 4 times this if $a+b$ is odd.
- Fibonacci proved that a congruum must be divisible by 24 and he also showed that for x, c such that $x^2 + c$ and $x^2 - c$ are both squares, then c is a congruum.
- He also proved that a square cannot be a congruum.

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Fibonacci Numbers

- The Fibonacci Quarterly is a modern journal devoted to studying mathematics related to this sequence.
- Solved Problems:
 - The only square Fibonacci numbers are 1 and 144!
 - The only cubic Fibonacci numbers are 1 and 8!
 - The only triangular Fibonacci numbers are 1, 3, 21 and 55!

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Fibonacci Numbers

- Unsolved Problems about Fibonacci numbers:
 - Are there infinitely many prime Fibonacci numbers?
 - Are 1, 8 and 144 the only powers that are Fibonacci numbers?

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Lucas Numbers

- **Edouard Lucas** (1842-1891) gave the name "Fibonacci Numbers" to the series written about by Leonardo of Pisa.
- He studied a second series of numbers which uses the same recurrence equation but starts with 2 and 1, that is, 2, 1, 3, 4, 7, 11, 18, ...
- These numbers are called **the Lucas numbers** in his honor.

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Edouard Lucas (1842-1891)

- Lucas is also well known for his invention of the **Tower of Hanoi** puzzle and other mathematical recreations.
- The Tower of Hanoi puzzle appeared in 1883 under the name of M. Claus.
- Notice that Claus is an anagram of Lucas!
- His four volume work on recreational mathematics *Récréations mathématiques* has become a classic.

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Nicole Oresme (1320-1382)

- A French priest and mathematician.
- He translated many of Aristotle's works and questioned many of the ideas which at that time were accepted without question.



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Nicole Oresme (1320-1382)

- Oresme was the greatest of the French writers of the 14th century.
- He wrote *Tractatus proportionum*, *Algorismus proportionum*, *Tractatus de latitudinibus formarum*, *Tractatus de uniformitate et difformitate intensionum*, and *Traité de la sphère*.
- In the *Algorismus proportionum* is the first use of fractional exponents.

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Nicole Oresme (1320-1382)

- In Tractatus de uniformitate, Oresme invented a type of coordinate geometry before Descartes, in fact, Descartes may have been influenced by his work.
- He proposed the use of a graph for plotting a variable magnitude whose value depends on another variable.

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Nicole Oresme (1320-1382)

- He was the first to prove Merton's Theorem, that is, that the distance traveled in a fixed time by a body moving under uniform acceleration is the same as if the body moved at uniform speed equal to its speed at the midpoint of the time period.
- He wrote Questiones Super Libros Aristotelis de Anima dealing with the nature, speed and reflection of light.

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Nicole Oresme (1320-1382)

- Oresme worked on infinite series and was the first to prove that the harmonic series $1 + 1/2 + 1/3 + 1/4 + \dots$ becomes infinite without bound, i.e., it diverges.
- In Livre du ciel et du monde, he opposed the theory of a stationary Earth as proposed by Aristotle and proposed rotation of the Earth some 200 years before Copernicus.
- He later rejected his own idea.

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Leibniz (1646 – 1716)

- Leibniz is considered to be one of the fathers of Calculus.
- We will discuss him in further detail later.
- For now, let's look at his work with infinite series.



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Leibniz (1646 – 1716)

- He discovered the following series for π :

$$\pi = 4 \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots \right)$$
- Leibniz's idea out of which his calculus grew was the inverse relationship of sums and differences for sequences of numbers.

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Repeating Decimals

- We can use a clever trick to determine the fraction equivalent to a given repeating decimal.
- For example, suppose we want to know what fraction is equal to 0.666666...
- Let x = the decimal.
- Then multiply both sides by 10.

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Repeating Decimals

- Finally, subtract the two equations.

$$\begin{array}{r} 10x = 6.6666666\ldots \\ x = 0.6666666\ldots \\ \hline 9x = 6 \end{array}$$

- Divide both sides by 9
- Thus, $x = 6/9 = 2/3$.

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Repeating Decimals

- If the pattern is two digits, you multiply by 100 instead of 10.
- For example, suppose we want to know what fraction is equal to $0.45454545\ldots$
- Let $x = 0.45454545\ldots$
- Multiply both sides by 100, so
- $100x = 45.45454545\ldots$

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Repeating Decimals

- Subtract the two equations:

$$\begin{array}{r} 100x = 45.45454545\ldots \\ x = 0.45454545\ldots \\ \hline 99x = 45 \end{array}$$

- The solve for x .
- Thus, $x = 45/99 = 5/11$.

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